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The Earth term in pulsar timing residuals is out of phase among the pulsars

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ABSTRACT

We aim to resolve a misunderstanding about whether the so-called “Earth term” in the pulsar timing response to a gravitational wave is in phase among a set of pulsars. We note that the misunderstanding has potentially arisen from the statements that the Earth term is “coherent” or “builds up coherently” among the pulsars. We clarify what authors mean by “coherent” in these statements, pointing out that “coherent” does not indicate that the Earth terms are in phase among the pulsars. Using the pulsar timing residuals induced by a continuous gravitational wave, we show that the Earth term does not align across different pulsars except for the special case when the gravitational-wave source is edge-on, i.e. when the orbital inclination angle of the source is either $\iota = \pi/2$ or $3\pi/2$. We demonstrate the same concept using the pulsar timing software `LIBSTEMPO` by plotting the Earth terms from a set of pulsars.

Keywords: gravitational waves — pulsars — Earth term — timing residuals

1. INTRODUCTION

The integral that accounts for the influence of a gravitational wave (GW) on the electromagnetic (EM) signal of the pulsar yields one piece at each of its endpoints: one that corresponds to the GW disturbance at the pulsar when the EM wave was emitted (the “pulsar term”), and one that corresponds to the GW disturbance at the Earth when the EM wave was received (the “Earth term”) (Lommen 2015). These two pieces in the residual response will in general be different in phase and frequency, and most GW detection strategies have to confront these differences. Many authors have made statements that for a given GW signal the Earth term is “coherent” or “builds up coherently” among a set of pulsars (Ellis et al. 2012a,b; Perrodin & Sesana 2018; Zhu et al. 2016; Babak et al. 2016; Lommen 2015; Pitkin 2012; Sesana & Vecchio 2010, to name only a few). The use of the word “coherent,” however, is ambiguous because these statements can be falsely interpreted as the Earth term being in phase across all the pulsars. Our goal in writing this paper is to resolve the misunderstanding these statements can cause.

There are several different meanings and uses of the word “coherent.” One is a description of the radiation itself, another is a description of the analysis. We speak of two or more waves being coherent if the waves have the same frequency and constant phase differences between them (Daintith 2009). For example,

$\sin(\omega t)$ and $\cos(\omega t)$ are coherent as they have the same frequency, ω , and a constant phase difference of $\pi/2$. Observe that coherent waves do not have to be in phase. However, waves with the same phase are often informally described as coherent, adding ambiguity in our use of the word.

“Coherent” can also be used to describe the way we form a statistic. We speak of an analysis or a sum being coherent when we add the waves (or any data) together before squaring the sum of the waves (Ellis et al. 2012b). Observe that this allows the waves to interfere with each other both constructively and destructively. This is in contrast to an “incoherent” sum in which we square each of the waves before adding them (Ellis et al. 2012b). Since each wave is squared, interference among the waves is not possible in incoherent analyses. We attempt to be very specific when we use the word “coherent” so as to avoid any ambiguity.

Many of the aforementioned authors utilize in their GW detection strategies the fact that the Earth term is “coherent” or “builds up coherently” among the pulsars. In this paper, we point out that “coherent” here does not imply that all the Earth terms have the same phase, and show that the Earth terms are generally out of phase among the pulsars. The Earth term is a combination of two terms: one is the contribution from the $+$ -polarization, and the other from the \times -polarization of the gravitational wave. The $+$ -polarization term is in phase across all the pulsars, and the \times -polarization term is in phase across all the pulsars. However, the combination of them is not, because the two terms have different relative amplitudes for different pulsars. This makes the Earth term have a different phase for each pulsar.¹

The paper is organized as follows. In Section 2, we review the mathematical expressions for the pulsar timing residuals induced by a continuous gravitational wave. In Section 3, we show that the Earth terms line up only in the two special cases when the orbital inclination angle of the source is either $\iota = \pi/2$ or $3\pi/2$. We demonstrate the same results by plotting the Earth terms of 19 different pulsars. Finally, we explain what authors mean by “coherent” and conclude in Section 4.

2. PULSAR TIMING RESIDUALS INDUCED BY A CONTINUOUS GRAVITATIONAL WAVE

We review the effect of a continuous gravitational wave emitted by a super-massive black hole binary (SMBHB) on the pulsar timing residuals. We assume that the SMBHB is non-spinning and in a circular orbit. We directly adopt the notations and calculations presented in Ellis et al. (2012b) (hereafter E12) and references therein. The gravitational wave perturbs the flat spacetime as it propagates, and the metric perturbation is expressed as:

$$h_{ab}(t) = h_+(t, \hat{\Omega})e_{ab}^+(\hat{\Omega}) + h_\times(t, \hat{\Omega})e_{ab}^\times(\hat{\Omega}), \quad (1)$$

where “ ab ” represents the indices of the tensor, $\hat{\Omega}$ the unit propagation vector pointing from the GW source to the solar system barycenter (SSB), and $h_{+,\times}$ the amplitudes of the $+$ - and \times -polarizations of the GW, respectively. The polarization tensors $e_{ab}^{+,\times}$ are given by

$$e_{ab}^+(\hat{\Omega}) = \hat{m}_a\hat{m}_b - \hat{n}_a\hat{n}_b, \quad (2)$$

$$e_{ab}^\times(\hat{\Omega}) = \hat{m}_a\hat{n}_b + \hat{n}_b\hat{m}_a. \quad (3)$$

¹ The situation is complicated by the fact that there are multiple definitions of the “Earth term” in the literature. Our statement that the Earth term is generally out of phase is true if we define the Earth term as in this paper, as in Equation (7). However, this statement does not hold for some definition of the Earth term. We discuss this in detail in Appendix A.

The unit vectors introduced are defined in $\{\hat{x}, \hat{y}, \hat{z}\}$ by

$$\hat{\Omega} = \{-\sin\theta \cos\phi, -\sin\theta \sin\phi, -\cos\theta\}, \quad (4)$$

$$\hat{m} = \{-\sin\phi, \cos\phi, 0\}, \quad (5)$$

$$\hat{n} = \{-\cos\theta \cos\phi, -\cos\theta \sin\phi, \sin\theta\}, \quad (6)$$

where θ, ϕ are respectively the polar and azimuthal angles of the GW source location. The coordinate system $\{\hat{x}, \hat{y}, \hat{z}\}$ is defined such that the North celestial pole is in the \hat{z} -direction, the vernal equinox is in the \hat{x} -direction, and $\hat{y} = \hat{z} \times \hat{x}$. Note that $\hat{m} \times \hat{n} = -\hat{\Omega}$ and that \hat{m} points to increasing right ascension α , and \hat{n} points to increasing declination δ .² Notice that we have two right-handed coordinate systems: $\{\hat{x}, \hat{y}, \hat{z}\}$ centered at the SSB and $\{\hat{m}, \hat{n}, -\hat{\Omega}\}$ at the location of the GW source.

To leading order and ignoring noise, the timing residual of a pulsar is written as:

$$r(t, \hat{\Omega}) = r^p(t, \hat{\Omega}) - r^e(t, \hat{\Omega}) \quad (7a)$$

$$= [F^+(\hat{\Omega})r_+(t_p) + F^\times(\hat{\Omega})r_\times(t_p)] - [F^+(\hat{\Omega})r_+(t) + F^\times(\hat{\Omega})r_\times(t)], \quad (7b)$$

where r^p (the first square-bracket) is the ‘‘pulsar term’’ and r^e (the second square-bracket) is the ‘‘Earth term.’’ Note that the Earth technically represents the SSB, as mentioned in E12. The antenna response functions $F^{+, \times}$, also known as the antenna pattern functions, of the given pulsar are defined as

$$F^+(\hat{\Omega}) = \frac{1}{2} \frac{(\hat{p} \cdot \hat{m})^2 - (\hat{p} \cdot \hat{n})^2}{1 + \hat{p} \cdot \hat{\Omega}}, \quad (8a)$$

$$F^\times(\hat{\Omega}) = \frac{(\hat{p} \cdot \hat{m})(\hat{p} \cdot \hat{n})}{1 + \hat{p} \cdot \hat{\Omega}}, \quad (8b)$$

where the unit vector \hat{p} points towards the pulsar from the Earth.

In Equation (7), t denotes the time of observation at the Earth, and t_p , which is called the pulsar time, is given by

$$t_p = t - L(1 + \hat{\Omega} \cdot \hat{p}) \equiv t - \tau. \quad (9)$$

The pulsar time t_p always precedes the time t at the Earth by τ , where $\tau = L(1 + \hat{\Omega} \cdot \hat{p})$ is the delay between two events: (1) the gravitational wave reaching the Earth, and (2) the information that the gravitational wave has reached the pulsar reaching the Earth (Lommen 2015). Thus the pulsar term r^p corresponds to an earlier epoch in the evolution of the GW source.³ However, notice that t_p does not exactly equal the time when the radio pulse was emitted from the pulsar. In fact, t_p equals the time of radio emission from the pulsar, i.e. $t_p = t - L$, only when $\hat{\Omega}$ is perpendicular to \hat{p} . If $\hat{\Omega} = \hat{p}$, for example, i.e. when the Earth is in between the pulsar and the GW source, then Equation (9) gives $t_p = t - 2L$.

Finally, $r_{+, \times}$ in Equation (7) are given by

$$r_+(t) = \frac{\mathcal{M}^{5/3}}{D\omega(t)^{1/3}} \left[-(1 + \cos^2 \iota) \cos 2\psi \sin 2(\Phi(t) - \phi_n) - 2 \cos \iota \sin 2\psi \cos 2(\Phi(t) - \phi_n) \right], \quad (10)$$

$$r_\times(t) = \frac{\mathcal{M}^{5/3}}{D\omega(t)^{1/3}} \left[-(1 + \cos^2 \iota) \sin 2\psi \sin 2(\Phi(t) - \phi_n) + 2 \cos \iota \cos 2\psi \cos 2(\Phi(t) - \phi_n) \right]. \quad (11)$$

² $\{\alpha, \delta\}$ and $\{\phi, \theta\}$ are related by $\phi = \alpha$ and $\theta = \pi/2 - \delta$.

³ To give an analogy, imagine you get your news by two sources: one by social media and one by a print newspaper delivered to your doorstep. The print newspaper is going to carry news from a time τ earlier, where τ represents the lag between information getting posted on social media and information arriving on your doorstep via the newspaper. If you look at social media and the print newspaper at the same time, your brain will be carrying news from two different times: news(now) + news(now - τ).

Here, ι is the inclination angle of the SMBHB orbital plane, ψ the GW polarization angle, and D the luminosity distance to the SMBHB. The chirp mass \mathcal{M} of the binary is defined as $\mathcal{M} = (m_1 m_2)^{3/5} / (m_1 + m_2)^{1/5}$, where m_1, m_2 represent the masses of the two black holes in the SMBHB. ϕ_n is the orbital phase at the line of nodes, which is defined as the intersection of the SMBHB orbital plane with the tangent plane of the sky (Wahlquist 1987). The orbital phase and frequency of the SMBHB are given respectively by

$$\Phi(t) = \Phi_0 + \frac{1}{32\mathcal{M}^{5/3}} \left(\omega_0^{-5/3} - \omega(t)^{-5/3} \right), \quad (12)$$

$$\omega(t) = \omega_0 \left(1 - \frac{256}{5} \mathcal{M}^{5/3} \omega_0^{8/3} t \right)^{-3/8}, \quad (13)$$

where Φ_0 and ω_0 are respectively the initial orbital phase and frequency determined at the time when the observation begins.⁴

It is important to note that different authors define their Earth term differently. As we will see in Section 3, the Earth term, r^e , does not generally align among the pulsars. However, there is another definition of the Earth term in the literature that does align across all the pulsars. We compare different definitions of the Earth term in the literature in Appendix A. In this paper, we follow the notations used in E12 and thus define r^e , given in Equation (7), as the Earth term.

3. THE EARTH TERMS ARE NOT IN PHASE

In this section, we show that the Earth term is generally out of phase among a set of pulsars. The Earth terms align only when the orbital inclination angle of the SMBHB is either $\iota = \pi/2$ or $3\pi/2$, i.e. when the source is edge-on. We demonstrate the same results by plotting the Earth terms of 19 different pulsars using the pulsar timing software `LIBSTEMPO` (Vallisneri 2020).⁵ The Earth term r^e can be fully written as (Equations 7, 10, and 11):

$$r^e(t, \hat{\Omega}) = -\frac{\mathcal{M}^{5/3}}{D\omega(t)^{1/3}} \left\{ F^+(\hat{\Omega}) \left[-(1 + \cos^2 \iota) \cos 2\psi \sin 2\Phi'(t) - 2 \cos \iota \sin 2\psi \cos 2\Phi'(t) \right] \right. \\ \left. + F^\times(\hat{\Omega}) \left[-(1 + \cos^2 \iota) \sin 2\psi \sin 2\Phi'(t) + 2 \cos \iota \cos 2\psi \cos 2\Phi'(t) \right] \right\}, \quad (14)$$

where $\Phi'(t) = \Phi(t) - \phi_n$. We simplify this equation employing the following trick. Since $\sin 2\Phi'(t)$ and $\cos 2\Phi'(t)$ are sinusoids of the same frequency $\omega(t)$ (Equation 13), the two terms inside the first square-bracket can be combined into a single sinusoid with some new amplitude $A(\iota, \psi)$ and new phase $\Phi_A(t)$. Notice that the frequency $\omega(t)$ remains the same, but the new phase $\Phi_A(t)$ depends on the ratio of the two coefficients $-(1 + \cos^2 \iota) \cos 2\psi$ and $-2 \cos \iota \sin 2\psi$. Using the same method, we combine the two terms inside the second square-bracket, producing a new amplitude $B(\iota, \psi)$ and a new phase $\Phi_B(t)$. This new phase $\Phi_B(t)$ is dependent on the ratio of the coefficients $-(1 + \cos^2 \iota) \sin 2\psi$ and $2 \cos \iota \cos 2\psi$. Then, Equation (14) is reduced to:

$$r^e(t, \hat{\Omega}) = -\frac{\mathcal{M}^{5/3}}{D\omega(t)^{1/3}} \left\{ F^+(\hat{\Omega}) A(\iota, \psi) \sin \Phi_A(t) + F^\times(\hat{\Omega}) B(\iota, \psi) \cos \Phi_B(t) \right\}. \quad (15)$$

Here, we see that the Earth term is a sum of two sinusoidal waves, $\sin \Phi_A(t)$ and $\cos \Phi_B(t)$, of the same frequency, $\omega(t)$. Therefore, the two terms in Equation (15) can *again* be combined into a single sinusoid

⁴ Φ_0 and ω_0 are related to the initial GW phase and frequency by $\Phi_0 = \Phi_{\text{gw},0}/2$ and $\omega_0 = \omega_{\text{gw},0}/2$.

⁵ The `LIBSTEMPO` library can be found at: <https://github.com/vallis/libstempo>.

with the same frequency, but with a new amplitude and a new phase that depends on the two coefficients $F^+(\hat{\Omega})A(\iota, \psi)$ and $F^\times(\hat{\Omega})B(\iota, \psi)$. However, note that $A(\iota, \psi)$ and $B(\iota, \psi)$ are the same for all the pulsars, and so are $\Phi_A(t)$ and $\Phi_B(t)$. The only things that differ between the pulsars are the antenna response functions $F^{+\times}$, which depend on the sky location, \hat{p} , of each pulsar (see Equation 8). This means that the overall phase of this single sinusoid, $r^e(t, \hat{\Omega})$, will be determined by the relative amplitudes of $F^+(\hat{\Omega})$ and $F^\times(\hat{\Omega})$. Hence, it follows that the Earth terms will have different phases for different pulsars.

There are two special cases when the Earth terms line up amongst the pulsars. This is when the second terms in both square-brackets in Equation (14) go to zero, i.e. when $\iota = \pi/2$ or $3\pi/2$ (when the SMBHB is edge-on). In these cases, we have $\cos \iota = 0$, and Equation (14) becomes:

$$\begin{aligned} r^e(t, \hat{\Omega}) \Big|_{\iota=\pi/2, 3\pi/2} &= \frac{\mathcal{M}^{5/3}}{D\omega(t)^{1/3}} \left\{ F^+(\hat{\Omega}) \cos 2\psi \sin 2\Phi'(t) + F^\times(\hat{\Omega}) \sin 2\psi \sin 2\Phi'(t) \right\} \\ &= \frac{\mathcal{M}^{5/3}}{D\omega(t)^{1/3}} \left(F^+(\hat{\Omega}) \cos 2\psi + F^\times(\hat{\Omega}) \sin 2\psi \right) \sin 2\Phi'(t). \end{aligned} \quad (16)$$

Thus, when $\iota = \pi/2$ or $3\pi/2$, the Earth terms from all the different pulsars will have the same phase, $2\Phi'(t)$, though their amplitudes will be different because the factor $F^+(\hat{\Omega}) \cos 2\psi + F^\times(\hat{\Omega}) \sin 2\psi$ varies among the pulsars due to $F^{+\times}$.

One might wonder if, for an arbitrary ι , we can choose a ψ such that all the Earth terms line up. However, there is no such value of ψ . For all the Earth terms to align, we need a condition where the phase of the Earth term does not depend on the ratio of F^+ and F^\times , as in Equation (16). This condition is achieved only when the $\sin 2\Phi'(t)$ terms in both square-brackets go to zero, or when the $\cos 2\Phi'(t)$ terms in both square-brackets vanish in Equation (14). To eliminate any of the terms, we need to choose $\psi = 0, \pi/2, \pi, 3\pi/2, \text{ or } 2\pi$. However, observe that none of these makes the Earth terms align amongst the pulsars. To see this clearly, consider the following example. When $\psi = 0$, the $\cos 2\Phi'(t)$ term vanishes from the first square-bracket and the $\sin 2\Phi'(t)$ term vanishes from the second square-bracket. So Equation (14) becomes:

$$r^e(t, \hat{\Omega}) \Big|_{\psi=0} = \frac{\mathcal{M}^{5/3}}{D\omega(t)^{1/3}} \left\{ F^+(\hat{\Omega})(1 + \cos^2 \iota) \cos 2\psi \sin 2\Phi'(t) - F^\times(\hat{\Omega})(2 \cos \iota) \cos 2\psi \cos 2\Phi'(t) \right\}. \quad (17)$$

Notice that we have basically obtained the same situation as in Equation (15). If we combine the two terms in Equation (17), the phase of the resultant sinusoid will depend on the ratio of $F^+(\hat{\Omega})$ and $F^\times(\hat{\Omega})$, making the Earth terms have a different phase for each pulsar. We acquire similar results for $\psi = \pi/2, \pi, 3\pi/2, \text{ and } 2\pi$. Hence, we conclude that there is no value of ψ that makes the Earth terms line up across all the pulsars.

We demonstrate our results using `LIBSTEMPO`'s function `add_cgw` with a set of 19 different pulsars.⁶ The function `add_cgw` in the `toasim` package injects a continuous GW emitted by a SMBHB into the timing residuals. It offers the option to exclude the pulsar term (by setting the parameter `psrTerm=False`), allowing us to extract only the Earth term from the timing residuals. In Figure 1, the Earth terms from 19 different pulsars are plotted for random values of ι and ψ . We clearly see that the Earth terms are out of phase among the pulsars. On the other hand, when the inclination angle is either $\iota = \pi/2$ or $3\pi/2$ (when the system is edge-on), all the Earth terms line up, as shown in Figure 2. Notice that the Earth terms have different amplitudes for different pulsars due to the factor $F^+(\hat{\Omega}) \cos 2\psi + F^\times(\hat{\Omega}) \sin 2\psi$ in Equation (16). Also, the Earth terms for some pulsars have an overall negative sign from other Earth terms. This is because $F^{+\times}$ can be either positive or negative depending upon each pulsar's direction \hat{p} (see Equations 8 and 16).

⁶ The `LIBSTEMPO` library can be found at: <https://github.com/vallis/libstempo>.

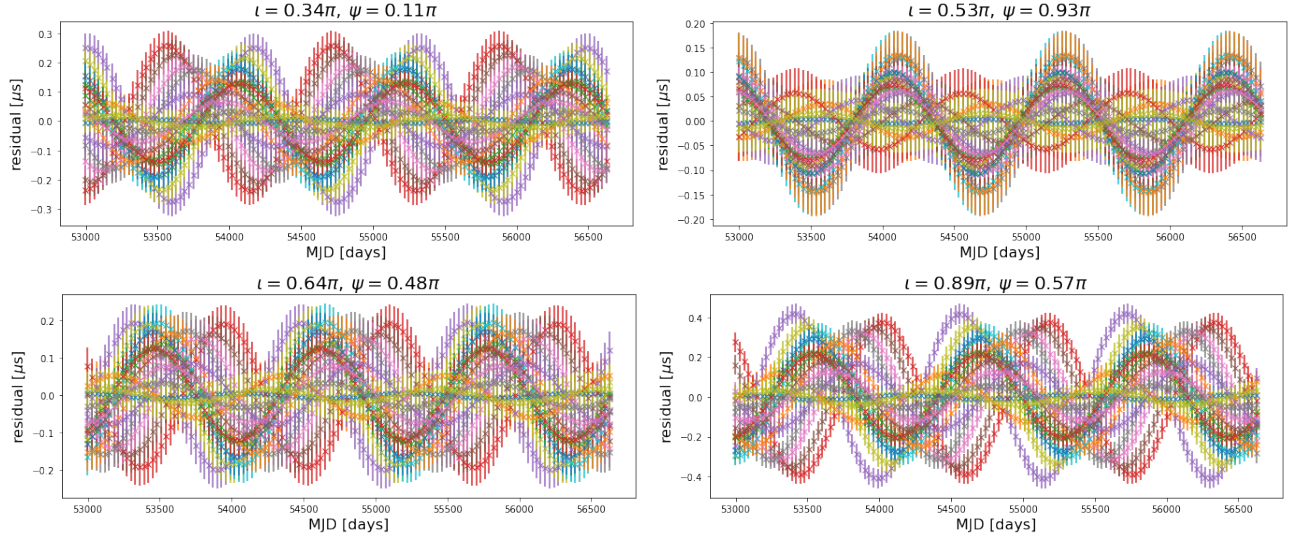


Figure 1. The Earth terms from 19 different pulsars with four different random inclination (ι) and polarization (ψ) angles. Each color represent a single pulsar. The Earth terms are clearly out of phase among the pulsars. However, the Earth terms do align when $\iota = \pi/2$ or $3\pi/2$, i.e. when the source is edge-on, as shown in Figure 2.

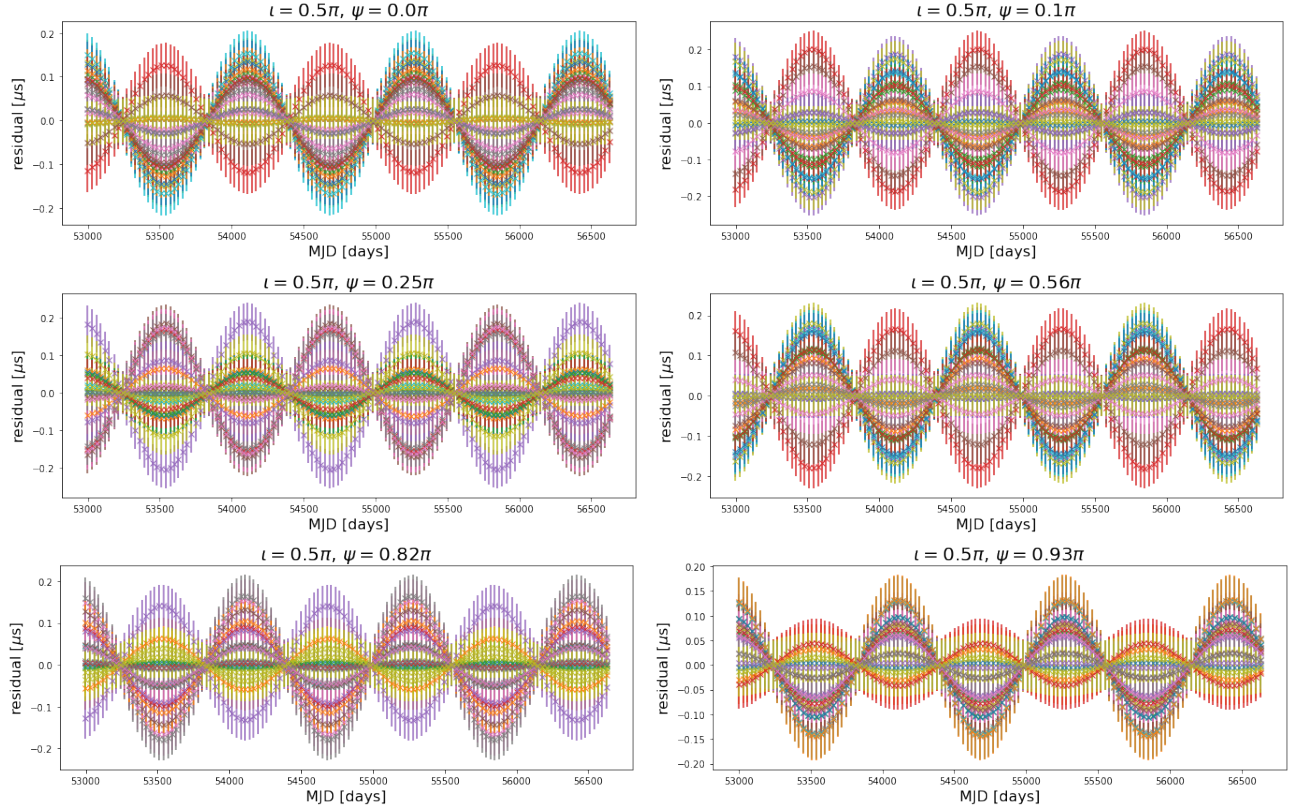


Figure 2. The Earth terms from 19 different pulsars with the inclination angle of $\pi/2$ (edge-on) and six different random polarization angles. Each color represent a single pulsar. This is one of the two special cases ($\iota = \pi/2$ or $3\pi/2$) where the Earth terms line up across all the pulsars. General cases when $\iota \neq \pi/2, 3\pi/2$ are shown in Figure 1. Note that the Earth terms for some pulsars have opposite signs because $F^{+, \times}$ can produce an overall negative sign depending on the pulsar location. The Earth terms have different amplitudes for different pulsars because $F^{+, \times}$ vary among the pulsars (see Equation 16).

4. CONCLUSION

We pointed out that our ambiguous use of the word “coherent” could lead to the misconception that the Earth terms in the pulsar timing residuals are in phase across all the pulsars. As we showed in Section 3, however, the Earth terms are generally out of phase.

The difference in phase among the Earth terms arises from the antenna response functions, $F^{+\times}$, which depend on the position, \hat{p} , of a particular pulsar. The Earth term is a combination of two sinusoidal terms: one corresponding to the + polarization which is multiplied by $F^+(\hat{\Omega})$ and the other corresponding to the \times -polarization which is multiplied by $F^\times(\hat{\Omega})$. These two terms have the same frequency, so we can combine them into a single Earth-term sinusoid which has the same frequency but a new amplitude and a new phase. The phase of this resultant sinusoid is dependent on the relative amplitudes of the two original sinusoidal terms. In particular, the new phase depends on the ratio of $F^+(\hat{\Omega})$ and $F^\times(\hat{\Omega})$, but $F^{+\times}(\hat{\Omega})$ vary among the pulsars. Therefore, the Earth term has a different phase for each pulsar. The Earth terms align only when the inclination of the SMBHB orbital plane is either $\iota = \pi/2$ or $3\pi/2$, i.e. when the binary system is edge-on. We demonstrated our results by plotting the Earth terms from 19 different pulsars, using the pulsar timing software `LIBSTEMPO` (see Figures 1 and 2).

A few GW detection techniques, for example the frequentist Earth-term \mathcal{F}_e -statistic, exploit the fact that the Earth term in the pulsar timing response to a gravitational wave is “coherent” or “builds up coherently” among a set of pulsars (e.g. E12; Babak et al. 2016, hereafter B16). We stress that the word “coherent” in these statements does not imply that the Earth term is in phase among the pulsars. In fact, Earth terms being “coherent” and Earth terms “building up coherently” mean two different things. The former is a description of the radiation, i.e. the Earth-term wave itself, whereas the latter is a description of the analysis used in the \mathcal{F}_e -statistic (see Section 1 for different meanings of the word “coherent”).

In the maximum likelihood calculation for \mathcal{F}_e in E12 and B16, the initial phase, Φ_0 , of the binary system (Equation 12) is one of the four parameters, $(\mathcal{M}^{5/3}D^{-1}, \iota, \Phi_0, \psi)$, over which the authors analytically maximize their statistic in order to search for a GW signal. By Earth terms being “coherent,” E12 and B16 mean that the parameter that refers to the initial phase, Φ_0 , of the binary is the same in all the Earth terms such that one needs to maximize over only one phase parameter for all the different pulsars. On the other hand, when the authors say that the Earth terms “build up coherently” among the pulsars, they are describing the way the Earth-term signals are combined in the calculation of the \mathcal{F}_e -statistic. The authors are making the point that the \mathcal{F}_e -statistic is a coherent statistic because in doing the analysis the Earth terms are summed before they are squared (as opposed to squared before they are summed). We refer the reader to E12 for complete mathematical treatment of the \mathcal{F}_e -statistic.

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APPENDIX

A. DIFFERENT DEFINITIONS OF THE EARTH TERM

In Section 3, we showed that the Earth term in the pulsar timing residuals is generally out of phase across the pulsars. However, there are multiple definitions of the “Earth term” in the literature, and whether the Earth term is in phase depends upon one’s definition of the Earth term. In this Appendix, we compare two

different definitions of the Earth term used in the literature and show that the statement that the Earth terms are generally out of phase does not hold true for the latter definition.

Some authors incorporate the antenna response functions $F^{+,\times}$ (Equation 8) in their Earth term so that the Earth term represents one component of the timing residual that is actually measured at the Earth (ignoring noise), as defined in this paper. Many authors including E12, B16, Corbin & Cornish (2010), Perrodin & Sesana (2018), and Sesana & Vecchio (2010) have adopted this definition. This is the Earth term r^e we use, given in Equation (7):

$$r^e(t, \hat{\Omega}) = -F^+(\hat{\Omega})r_+(t) - F^\times(\hat{\Omega})r_\times(t). \quad (\text{A1})$$

On the other hand, some define $r_{+,\times}(t)$ to be their Earth term (Equations 10 and 11). Note that $r_+(t)$ and $r_\times(t)$ are the ingredients in the calculation of r^e as shown in Equation (A1), but they do not yield a measurable quantity until they are multiplied by $F^{+,\times}$. A number of authors including Jenet et al. (2004) and Arzoumanian et al. (2020) have used this latter definition. Note that unlike r^e which is generally out of phase, each of $r_+(t)$ and $r_\times(t)$ is in phase across all the pulsars because $r_{+,\times}(t)$ depend solely on the properties of the GW source (e.g. ι , ψ , Φ , ϕ_n). Hence, the statement that the Earth term is generally out of phase amongst the pulsars does not hold true for this latter definition.

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