Memorandum 006

Transmission Functions for Polynomial Fits

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Fitting a polynomial model to a time series and subtracting it removes variance from the data at the lowest frequencies. How much variance is removed is quantified by the transmission function, $\mathcal{T}(f)$, which is the ratio of the variance remaining at frequency f to the initial variance at the same frequency. In the pulsar timing array context, this formalism was introduced Blandford et al. (1984), and further developed by Cordes (2013) and Madison et al. (2013). Here we will compute the transmission function for polynomial fits of various orders.

Let us consider the idealized case of data sampled continuously over a finite observing time, T. For convenience, we will set $t = 0$ at the mid-point of the observation, so that the data is represented by a function, $x(t)$, supported on the interval $[-T/2, T/2]$. Appropriately scaled versions of the Legendre polynomials $P_n(x)$ form a complete orthogonal basis for the space of (appropriately well-behaved) functions of this form. In particular, we have

$$\int_{-T/2}^{-T/2} P_m\left(\frac{2t}{T}\right) P_n\left(\frac{2t}{T}\right) dt = \frac{T\delta_{mn}}{2n+1}. \tag{1}$$

It follows that we can expand $x(t)$ in a series of the form

$$x(t) = \sum_{k=0}^{\infty} C_k P_k\left(\frac{2t}{T}\right), \tag{2}$$

where the coefficients C_k are given by

$$C_k = \frac{2k+1}{T} \int_{-T/2}^{-T/2} P_k\left(\frac{2t}{T}\right) y(t)\, dt. \tag{3}$$

The sum of the first n terms in this series is the polynomial of degree $n-1$ most closely approximating $x(t)$ in the least-squares sense – in other words, the result of least-squares fitting a polynomial of degree $n-1$ to $x(t)$.

Because fitting and subtracting a polynomial is linear in $x(t)$, we can consider its effect on each frequency separately, so we need only consider the case where $x(t) = e^{2\pi ift}$. In this case, the coefficiencts $C_k(f)$ are given by

$$C_k(f) = \frac{2k+1}{T} \int_{-T/2}^{-T/2} P_k\left(\frac{2t}{T}\right) e^{2\pi ift}\, dt. \tag{4}$$

The total variance of the original signal, $x(t) = e^{2\pi ift}$, is given by

$$\sigma_0^2(f) = \frac{1}{T} \int_{-T/2}^{T/2} \left|e^{2\pi ift}\right|^2 dt = 1, \tag{5}$$

whereas the variance of the approximating polynomial

$$\hat{x}(t) = \sum_{k=0}^{n} C_k(f) P_k\left(\frac{2t}{T}\right) \tag{6}$$

is given by

$$\sigma^2(f) = \frac{1}{T} \int_{-T/2}^{T/2} |\hat{x}(t)|^2 dt = \sum_{k=0}^{n} \frac{|C_k(f)|^2}{2k+1} \tag{7}$$

It follows that the transmission function, $\mathcal{T}_n(f)$, for fitting and subtracting a polynomial of degree n is

$$\mathcal{T}_n(f) = \frac{\sigma_0^2(f) - \sigma^2(f)}{\sigma_0^2(f)} = 1 - \sum_{k=0}^{n} \frac{|C_k(f)|^2}{2k+1}. \tag{8}$$

Equation (4) can be simplified by replacing t and f with the dimensionless variables $u = 2t/T$ and $v = \pi fT$, in which case it becomes

$$C_k(v) = \left(k + \frac{1}{2}\right) \int_{-1}^{1} P_k(u) e^{iuv}\, du, \tag{9}$$

In particular, we have

$$C_0(v) = \frac{1}{2}\int_{-1}^{1} e^{iuv}\,du = \frac{\sin v}{v}, \tag{10}$$

$$C_1(v) = \frac{3}{2}\int_{-1}^{1} ue^{iuv}\,du = \frac{3i(\sin v - v\cos v)}{v^2}, \tag{11}$$

$$C_2(v) = \frac{5}{4}\int_{-1}^{1} \big(3u^2-1\big)e^{iuv}\,du = \frac{5[(v^2-3)\sin v + 3v\cos v]}{v^3}. \tag{12}$$

Explicit expressions for $\mathcal{T}_n(v)$ for $n = 0, 1, 2$ may be obtained by substituting these results into equation (8). In particular,

$$\mathcal{T}_0(v) = 1 - \frac{\sin^2 v}{v^2}, \tag{13}$$

$$\mathcal{T}_1(v) = \mathcal{T}_0(v) - \frac{3(\sin v - v\cos v)^2}{v^4}, \tag{14}$$

$$\mathcal{T}_2(v) = \mathcal{T}_1(v) - \frac{5\big[(v^2-3)\sin v + 3v\cos v\big]^2}{v^6}. \tag{15}$$

Qualitatively, each of these transmission functions rises from $\mathcal{T} = 0$ at $v = 0$ to oscillate just under $\mathcal{T} = 1$ at large values of v. We can find the asymptotic behavior of the transmission function at low frequencies by expanding each $C_k(v)$ around $v = 0$. This gives

$$C_0(v) = 1 - \frac{v^2}{6} + \frac{v^4}{120} - \frac{v^6}{5040} + \mathcal{O}(v^8), \tag{16}$$

$$C_1(v) = v - \frac{v^3}{10} + \frac{v^5}{280} + \mathcal{O}(v^7), \tag{17}$$

$$C_2(v) = -\frac{v^2}{3} + \frac{v^4}{42} + \mathcal{O}(v^6). \tag{18}$$

Substituting these series expansions into equation (8), we obtain

$$\mathcal{T}_0(v) = \frac{v^2}{3} - \frac{2v^4}{45} + \frac{v^6}{315} + \mathcal{O}(v^8), \tag{19}$$

$$\mathcal{T}_1(v) = \frac{v^4}{45} - \frac{4v^6}{1575} + \mathcal{O}(v^8), \tag{20}$$

$$\mathcal{T}_2(v) = \frac{v^6}{1575} + \mathcal{O}(v^8). \tag{21}$$

This means that, near $f = 0$, we have

$$\mathcal{T}_0(f) \sim \frac{\pi^2 T^2}{3} f^2, \tag{22}$$

$$\mathcal{T}_1(f) \sim \frac{\pi^4 T^4}{45} f^4, \tag{23}$$

$$\mathcal{T}_2(f) \sim \frac{\pi^6 T^6}{1575} f^6. \tag{24}$$

One could continue this process to find the asymptotic behavior of $\mathcal{T}_n(f)$ for higher values of n, but it turns out there's a way to solve the problem for arbitrary n. First, note that $C_k(v) = \mathcal{O}(v^k)$ for each value of k considered so far. This is true in general: expanding the right-hand side of equation (25) in a power series in v gives

$$C_k(v) = \left(k + \frac{1}{2}\right) \sum_{n=0}^{\infty} \frac{(iv)^n}{n!} \int_{-1}^{1} u^n P_k(u)\, du, \tag{25}$$

but since $P_k(u)$ is orthogonal to all polynomials of degree less than k, all terms with $n < k$ are zero.

Since we can write

$$\sum_{k=0}^{\infty} C_k(v) P_k(u) = e^{iuv}, \tag{26}$$

it follows that

$$\begin{aligned} \sum_{k=0}^{n} C_k(v) P_k(u) &= e^{iuv} - \sum_{k=n+1}^{\infty} C_k(v) P_k(u) \\ &= e^{iuv} + \mathcal{O}(v^{n+1}) \\ &= \sum_{k=0}^{n} \frac{(iuv)^k}{k!} + \mathcal{O}(v^{n+1}). \end{aligned} \tag{27}$$

Up to terms of order v^{n+1}, each side of this equation is a polynomial of degree n in u. Equating the leading terms gives

$$\frac{1}{2^n} \binom{2n}{n} u^n C_n(v) = \frac{i^n u^n v^n}{n!} + \mathcal{O}(v^{n+1}), \tag{28}$$

where we have used a well-known expession for the leading coefficient of $P_n(u)$. It follows that

$$C_n(v) = \frac{(2iv)^n}{n!\binom{2n}{n}} + \mathcal{O}(v^{n+1}). \tag{29}$$

Computing the total variance on both sides of equation (26) gives

$$\sum_{k=0}^{\infty} \frac{|C_k(v)|^2}{2k+1} = 1. \tag{30}$$

Combining this with equation (8) gives

$$\mathcal{T}_n(v) = \sum_{k=n+1}^{\infty} \frac{|C_k(v)|^2}{2k+1} = \frac{|C_{n+1}(v)|^2}{2n+3} + \mathcal{O}(v^{n+2}). \tag{31}$$

It follows that, near $v = 0$,

$$\mathcal{T}_n(v) = \frac{(2v)^{2n+2}}{(2n+3)(n+1)!^2\binom{2n+2}{n+2}^2} + \mathcal{O}(v^{n+2}), \tag{32}$$

or, in terms of f,

$$\mathcal{T}_n(f) \sim \frac{(2\pi T)^{2n+2}}{(2n+3)(n+1)!^2\binom{2n+2}{n+1}^2} f^{2n+2}. \tag{33}$$

Substituting $n = 0, 1, 2$ into equation (33) reproduces the results of equations (22), (23), and (24). For $n = 3, 4$ we find

$$\mathcal{T}_3(f) \sim \frac{\pi^8 T^8}{99\,225} f^8, \tag{34}$$

$$\mathcal{T}_4(f) \sim \frac{\pi^{10} T^{10}}{9\,823\,275} f^{10}. \tag{35}$$

$$\tag{36}$$

It is also possible to relax the assumption that the data are sampled continously and uniformly, at least for $\mathcal{T}_0(f)$ and $\mathcal{T}_1(f)$. Suppose that the data are instead sampled at N discrete times t_i. For convenience, we can assume that the mean of the sample times is zero:

$$\frac{1}{N} \sum_{i=1}^{N} t_i = 0. \tag{37}$$

Let σ^2, γ, and κ be the variance, skewness, and kurtosis of the sample times, i.e.,

$$\frac{1}{N}\sum_{i=1}^{N} t_i^2 = \sigma^2, \tag{38}$$

$$\frac{1}{N}\sum_{i=1}^{N} t_i^3 = \gamma\sigma^3, \tag{39}$$

$$\frac{1}{N}\sum_{i=1}^{N} t_i^4 = \kappa\sigma^4. \tag{40}$$

We can define an inner product between functions sampled at the times t_i by

$$\langle x(t), y(t)\rangle = \frac{1}{N}\sum_{i=1}^{N} x(t_i)y(t_i). \tag{41}$$

The three polynomials

$$p_0(t) = 1, \tag{42}$$

$$p_1(t) = t, \tag{43}$$

$$p_2(t) = t^2 - \gamma\sigma t - \sigma^2 \tag{44}$$

are orthogonal with respect to this inner product; that is, $\langle p_m(t), p_n(t)\rangle = 0$ whenever $m \neq n$. They also satisfy

$$\left\langle p_0(t)^2 \right\rangle = 1, \tag{45}$$

$$\left\langle p_1(t)^2 \right\rangle = \sigma^2, \tag{46}$$

$$\left\langle p_2(t)^2 \right\rangle = (\kappa - \gamma^2 - 1)\sigma^4. \tag{47}$$

Just as in the continuously sampled case, we can expand an arbitrary function $x(t)$ in terms of these polynomials, writing

$$x(t) = c_0 p_0(t) + c_1 p_1(t) + c_2 p_2(t) + r(t), \tag{48}$$

where the remainder $r(t)$ is orthogonal to each of the previous terms, and the coefficients are given by $c_k = \langle x(t), p_k(t)\rangle$. As before, to compute the transmission function, we must consider $x(t) = e^{2\pi i f t}$. In this case, the coefficients are given by

$$c_0(f) = \frac{1}{N} \sum_{j=1}^{N} e^{2\pi i f t_j}, \tag{49}$$

$$c_1(f) = \frac{1}{N} \sum_{j=1}^{N} t_j e^{2\pi i f t_j}, \tag{50}$$

$$c_2(f) = \frac{1}{N} \sum_{j=1}^{N} \left(t_j^2 - \gamma\sigma t_j - \sigma^2\right) e^{2\pi i f t_j}. \tag{51}$$

The full transmission functions $\mathcal{T}_0(f)$, $\mathcal{T}_1(f)$, and $\mathcal{T}_2(f)$ can be written

$$\mathcal{T}_0(f) = 1 - |c_0(f)|^2, \tag{52}$$

$$\mathcal{T}_1(f) = 1 - |c_0(f)|^2 - \frac{|c_1(f)|^2}{\sigma^2}, \tag{53}$$

$$\mathcal{T}_2(f) = 1 - |c_0(f)|^2 - \frac{|c_1(f)|^2}{\sigma^2} - \frac{|c_2(f)|^2}{(\kappa - \gamma^2 - 1)\sigma^4}, \tag{54}$$

but cannot be expressed in any simpler form. However, in the low frequency limit, the expressions do simplify somewhat. In particular, expanding $e^{2\pi i f t}$ in a power series around $f = 0$, we have

$$\begin{aligned} c_0(t) &= \frac{1}{N} \sum_{j=1}^{N} \left(1 + 2\pi i f t_j - 2\pi^2 f^2 t_j^2 - \frac{4\pi^3}{3} i f^3 t_j^3 + \frac{2\pi^4}{3} f^4 t_j^4 + \mathcal{O}(f^5)\right) \\ &= 1 - 2\pi^2 \sigma^2 f^2 - \frac{4\pi^3}{3} i \gamma \sigma^3 f^3 + \frac{2\pi^4}{3} \kappa \sigma^4 f^4 + \mathcal{O}(f^5), \end{aligned} \tag{55}$$

$$\begin{aligned} c_1(t) &= \frac{1}{N} \sum_{j=1}^{N} \left(t_j + 2\pi i f t_j^2 - 2\pi^2 f^2 t_j^3 - \frac{4\pi^3}{3} i f^3 t_j^4 + \mathcal{O}(f^4)\right) \\ &= 2\pi i \sigma^2 f - 2\pi i \gamma \sigma^3 f^2 - \frac{4\pi^3}{3} i \kappa \sigma^4 f^3 + \mathcal{O}(f^4). \end{aligned} \tag{56}$$

Substituting these results into equations (52) and (53) gives

$$\mathcal{T}_0(f) \sim 4\pi^2 \sigma^2 f^2, \tag{57}$$

$$\mathcal{T}_1(f) \sim 4\pi^4 \left(\kappa - \gamma^2 - 1\right) \sigma^4 f^4. \tag{58}$$

Using the variance $\sigma^2 = T^2/12$, skewness $\gamma = 0$, and kurtosis $\kappa = 9/5$ for a uniform distribution in these expressions reproduces equations (22) and (23). A similar result is possible for $\mathcal{T}_2(f)$, showing that it is asymptotically proportional to f^6, but with a much more complicated coefficient.

References

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